Theory of Coherent Tunneling in Exciton Condensates of Bilayer Quantum Hall Systems

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Due to strong interlayer correlations, the bilayer quantum Hall system is a single coherent system as a whole rather than a weakly-coupled set of two independent systems, which makes conventional tunneling theories inapplicable. In this paper, we develop a theory of interlayer tunneling in coherent exciton condensates of bilayer quantum Hall systems at total filling factor $\nu_T = 1$. One of the most important consequences of our theory is that the zero-bias interlayer tunneling conductance peak is strongly enhanced, but fundamentally finite even at zero temperature. We explicitly compute the height of the conductance peak as a function of interlayer distance, which is compared with experiment. It is emphasized that the interlayer distance dependence of the conductance peak is one of the key properties distinguishing between the spontaneous coherence due to many-body effects of the Coulomb interaction and the induced coherence due to the single-particle tunneling gap. It is also emphasized that, though the strongly enhanced tunneling conductance originates from the interlayer phase coherence, it is not the usual Josephson effect. We propose an experimental setup for the true Josephson effect in couterflowing current measurements for a coupled set of two bilayer quantum Hall systems, which is a more precise analogy with the real Josephson effect in superconductivity.

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I. INTRODUCTION

Strongly-enhanced interlayer conductance peaks observed by Spielman *et al.*[1] near zero bias in bilayer quantum Hall systems at total filling factor $\nu_T = 1$ have attracted much interest for many reasons. One of the main reasons is the existence of a rather precise analogy between the bilayer quantum Hall effect[2] and superconductivity. That is, the ground state of the bilayer quantum Hall effect at small interlayer distance $d/l_B \ll 1$ ($l_B = \sqrt{\hbar c/eB}$) can be mapped onto the BCS-type wavefunction describing the Bose-Einstein condensate of exciton pairs formed between electrons and holes residing across the interlayer barrier.

Bose-Einstein condensation of neutral excitons has, in fact, been sought after in semiconductors for decades. In particular, there have been fascinating recent experiments on possible condensation of optically generated indirect excitons[3], for which, however, evidence is not yet conclusive. On the other hand, in the bilayer quantum Hall effect, it is generally accepted that the strongly-enhanced conductance peak is a direct indication of the macroscopic phase coherence.

To be concrete with respect to the mapping between superconductivity and the bilayer quantum Hall effect, let us begin by writing the ground state wave function at $d/l_B \to 0$. The explicit ground state wave function at $d/l_B \to 0$ is very instructive in illustrating the essential physics even though the ground state is substantially more complicated for general d/l_B, especially near the critical d/l_B where the exciton condensate disappears. (Note that we will use numerical methods to study the ground state at general d/l_B.) The ground state wave function at $d/l_B \to 0$ is known as the Halperin's (1,1,1) state[4]:

$$|\psi_{111}\rangle = \prod_m (c^\dagger_{m\uparrow} + c^\dagger_{m\downarrow})|0\rangle, \tag{1}$$

where m is a momentum index of the lowest Landau level. In the above, the pseudospin representation is used: $\uparrow$ and $\downarrow$ indicate the top and the bottom layer, respectively. It is important to note that, contrary to the usual representation of the (1,1,1) state,

$$\psi_{111} = \prod_{i,j\in\uparrow}(z_i - z_j)\prod_{k,l\in\downarrow}(z_k - z_l)\prod_{m\in\uparrow,n\in\downarrow}(z_m - z_n) \tag{2}$$

which denotes only orbital components, Eq.(1) describes the full wave function including both the orbital and the layer degree of freedom[5].

The ground state wave function described by Eq.(1) has an isomorphic structure to the BCS wave function, which can be made more apparent after the following reorganization:

$$\begin{aligned}|\psi_{111}\rangle &= \prod_m (1 + c^\dagger_{m\uparrow} c_{m\downarrow}) \prod_{m'} c^\dagger_{m'\downarrow}|0\rangle \\ &= \prod_m (1 + c^\dagger_{m\uparrow} c_{m\downarrow})|\text{new vacuum}\rangle \\ &= \prod_m (1 + c^\dagger_{m\uparrow} h^\dagger_{m\downarrow})|\text{new vacuum}\rangle, \end{aligned} \tag{3}$$

where $h^\dagger$ is a creation operator for holes, acting on the fully-filled bottom layer. Because of this analogy, it is

rather natural to expect that the bilayer quantum Hall state may have a coherence in the phase associated with interlayer electron-number difference, which is similar to the phase coherence associated with the total number of Cooper pairs in superconductivity. The phase coherence between states with different Cooper-pair numbers is the origin of the Josephson effect in superconductivity. Naturally, this similarity led many previous authors[6,7] to predict the Josephson effect in bilayer quantum Hall systems. So, in this context, the strongly enhanced conductance observed by Spielman *et al.*[1] seemed to be exactly the experimental verification needed. There are, however, some important properties of the conductance peak indicating that this phenomenon is not the conventional Josephson effect; most notably, experiments suggest that the height and the width of the zero-bias conductance peaks saturate to finite values when available finite-temperature data are extrapolated to the zero temperature limit[8,9]. This means that there is no DC current strictly at zero bias voltage in contrast to the real Josephson effect in superconductivity.

This apparent discrepancy gave rise to two groups of thought. In one group, the enhanced conductance is still regarded as DC Josephson effect, but its height is reduced by complicated disorder-induced fluctuations[10,11,12,13]. On the other hand, others[14] have argued that there is no exact analog of the Josephson effect in the experimental setup measuring interlayer tunneling currents in a single set of bilayer quantum Hall systems. It is because the bilayer system as a whole is a single Bose-Einstein condensate(BEC), not a set of two independent BEC's. While this argument itself is clearly true, it is still necessary to explain why and how the enhanced interlayer conductance is related to the BEC of excitons. The resolution to this crucial issue has remained elusive even after a very extensive body of research efforts continuing to recent years[15,16,17,18,19,20].

Important issues to be addressed are, in particular, (i) exactly what is the process of coherent tunneling in interlayer tunneling current measurements, (ii) how this process can be mathematically formulated, and finally (iii) what is the precise relationship between the interlayer phase coherence and the zero-bias conductance. It is the goal of this paper[21] to address these questions. Specifically, we would like to provide a quantitative analysis of the zero-bias conductance peak as a function of d/l_B, which, in turn, generates a sharp distinction between the coherent and incoherent tunneling processes.

The d/l_B dependence of the conductance peak is important also because it distinguishes between the coherence due to many-body effects of the Coulomb interaction and that due to the single-particle tunneling. In this paper, we are primarily interested in the regime of sufficiently small single-particle tunneling, $t/(e^2/\epsilon l_B) \ll 1$. We are interested in this regime mainly because experimental values of $t/(e^2/\epsilon l_B)$ are really quite small; in recent experimental setups, it can be as small as $10^{-6} - 10^{-7}$. This limit is also quite interesting in a theoretical point of view since the interlayer phase coherence is induced purely by many-body effects of the Coulomb interaction in the absence of single-particle tunneling. In later sections, we will investigate similarities and differences between the spontaneous coherence caused by the Coulomb interaction and the induced coherence caused by single-particle tunneling.

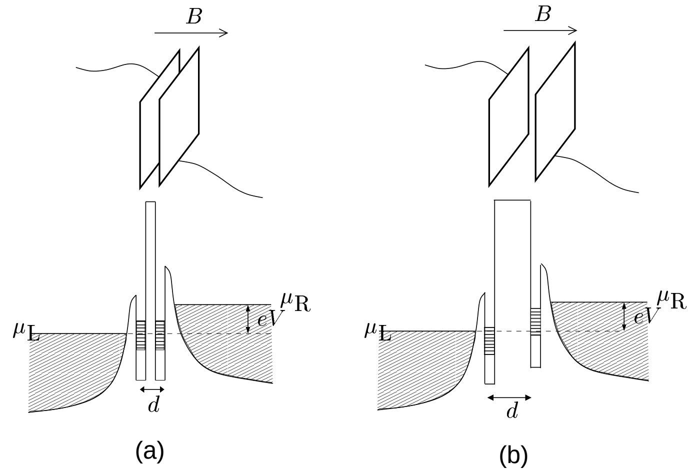


FIG. 1: Diagram showing the difference between (a) coherent tunneling and (b) incoherent tunneling. Note that, for the coherent ground state, there is no chemical potential difference between the two layers, as depicted in (a). When d is sufficiently large as shown in (b), however, each layer becomes independent, and therefore the interlayer coherence is lost.

This paper is organized as follows; we begin in Section II by distinguishing physical situations for coherent tunneling from those for incoherent one. We provide a physical picture as to exactly how interlayer tunneling can be enhanced in the presence of coherence. We, then, present our quantitative analysis in Section III by providing a mathematical derivation of the many-body tunneling Hamiltonian and its connection to the interlayer tunneling conductance. In Section IV, we numerically evaluate the coherent tunneling conductance as a function of d/l_B and compare it with experiment. In Section V, we propose an experimental setup for the Josepson effect in counterflowing current measurements for a coupled set of two bilayer quantum Hall systems. We finally conclude in Sec. VI.

II. COHERENT TUNNELING VERSUS INCOHERENT TUNNELING

As mentioned in the introduction, the bilayer quantum Hall system itself is a single superfluid system. The very existence of the excitation gap required for the quantized Hall resistance indicates that the ground state wave function is robust against small perturbations, in which case the bilayer system can be assumed to maintain its equilibrium. External perturbations such as bias voltage, then, can be taken as perturbations to the bilayer system

as a whole, not to the individual layers. In this situation, there is no chemical potential difference between the two layers, as depicted in Fig.1 (a).

In the situation described by Fig.1 (a), tunneling is coherent because the aforementioned robustness of the ground state is due to the interlayer coherence. Physically speaking, coherent interlayer tunneling is obtained as follows: Imagine that an electron is inserted into the top layer of the bilayer quantum Hall system, as depicted in the top diagram in Fig.2. The inserted electron, then, becomes a quasiparticle on top of the ground state with an energy penalty (the second diagram from the top in Fig.2). The energy penalty can be caused either by the Coulomb interaction (for the spontaneous coherence) or by the single-particle tunneling gap (for the induced coherence). The so-introduced quasiparticle, then, immediately becomes a part of the coherent ground state by resonating between the two layers (the third diagram from the top). The electron is finally ejected from the bottom layer, and thereby the interlayer current flows. It is important to note that, when there is an interlayer coherence, tunneling can be strongly enhanced because all inserted electrons will tunnel in a synchronized fashion.

The situation is quite different when there is no interlayer coherence. There are two important cases where the interlayer coherence can be lost; (i) for a fixed filling factor, namely $\nu_T = 1$ in our case, d/l_B becomes large or (ii) simply $B = 0$ in which interactions play a weaker role. For the first case, each layer forms an individual composite Fermi sea[22,23,24] at filling factor $\nu = 1/2$. What actually tunnel between the layers, however, are real electrons rather than composite fermions. In this situation, the interlayer tunneling is suppressed due to the fact that the composite Fermi sea is a highly correlated state so that the sudden insertion of an uncorrelated electron requires a large energy cost.

When $B = 0$, the situation is simpler: ordinary electrons tunnel. In this situation, the interlayer tunneling current can be evaluated in a conventional perturbation theory assuming that the single-particle tunneling gap, t, is small and therefore the two layers become independent and have individual chemical potentials, as depicted in Fig.2 (b). To be concrete, let us start with the tunneling Hamiltonian, H_t:

$$H_t \;=\; t\sum_{\mathbf{k}}(c^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}\downarrow} + c^\dagger_{\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}) \tag{4}$$

where t is the single-particle tunneling gap and the pseudospin representation is used. It is assumed that momenta parallel to the two-dimensional plane are conserved; in other words, there is no tunneling between different $\mathbf{k}$'s. In this representation, it is not too difficult to show that the tunneling current operator, $\hat{J}$, is given as follows:

$$\hat{J} \;=\; eti\sum_{\mathbf{k}}(c^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}\downarrow} - c^\dagger_{\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}) \tag{5}$$

When t is small, one can use a conventional first-order S-matrix expansion to compute the expectation value of

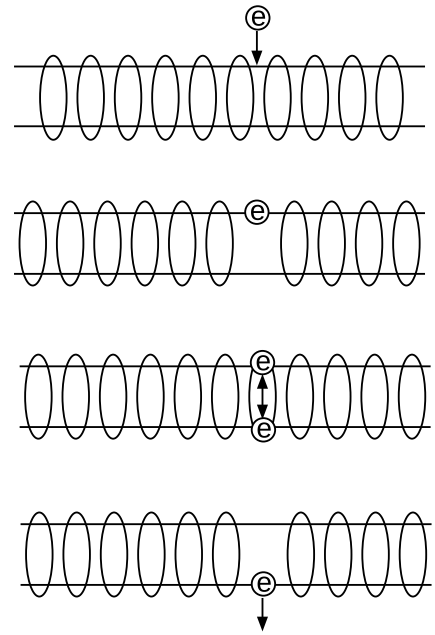


FIG. 2: Schematic diagrams for coherent interlayer tunneling. Note that each ellipse indicates the resonance between the top and the bottom layer. Imagine that an electron is inserted into the top layer of the bilayer quantum Hall system, as depicted in the top diagram. The inserted electron, then, becomes a quasiparticle on top of the ground state with an energy penalty (the second diagram from the top). The so-introduced quasiparticle, then, immediately becomes a part of the coherent ground state by resonating between the two layers (the third diagram from the top). It is important to note that the interlayer resonance can be caused either by many-body effects of the Coulomb interaction or by the single-particle tunneling. Finally, the electron is ejected from the bottom layer, and thereby the interlayer current flows (the bottom diagram).

the current operator:

$$I(t) = -i\int_{-\infty}^{t} dt'\langle[\hat{J}(t), H_T(t')]\rangle, \tag{6}$$

which, after some algebra, leads to the following expression:

$$I = 2et^2\int\frac{d\varepsilon}{2\pi}\sum_{\mathbf{k}}\rho_{\mathbf{k}\uparrow}(\varepsilon)\rho_{\mathbf{k}\downarrow}(\varepsilon+eV)[f(\varepsilon)-f(\varepsilon+eV)] \tag{7}$$

where $\rho_{\mathbf{k}}(\varepsilon)$ is the density of states for a given $\mathbf{k}$ (which is also known the spectral function), and $f(\varepsilon)$ is the usual Fermi-Dirac distribution function. Note that $\rho_{\mathbf{k}\uparrow}(\varepsilon) = \rho_{\mathbf{k}\downarrow}(\varepsilon) \equiv \rho_{\mathbf{k}}(\varepsilon)$ because of symmetry. Now, since the z-component motion of electrons is frozen due to the confinement to the two-dimensional plane, $\rho_{\mathbf{k}}(\varepsilon)$ is sharply peaked around its center (which depends on $\mathbf{k}$). As a consequence, the integrand in Eq.(7) vanishes except for small eV:

$$\rho_{\mathbf{k}}(\varepsilon)\rho_{\mathbf{k}}(\varepsilon+eV) \simeq \rho^2_{\mathbf{k}}(\varepsilon)e^{-(eV)^2/2\delta^2}, \tag{8}$$

where the size of δ is determined by the static disorder, which controls the width of the quasiparticle coherence peak. For small δ, as is expected in high-mobility 2D systems of interest, this single-particle tunneling peak is almost a delta-function with a small resonance width $eV \simeq \delta$.

This resonance behavior of the interlayer tunneling conductance is, in fact, observed at $B=0$ in the same sample[1] showing the enhancement of the zero-bias tunneling conductance at $\nu_T = 1$ for small d/l_B. So, at least superficially, incoherent tunneling seems to generate similar physical consequences as coherent tunneling. There are, however, indications suggesting that the $B=0$ conductance peak is actually different from the $\nu_T = 1$ peak. One indication is that the $B=0$ conductance peak is 100 times smaller than the peak at $\nu_T = 1$. Furthermore, the $B=0$ peak is at zero bias only when the two layers have the same density. If the densities are different, the $B=0$ peak shifts accordingly. In contrast, the $\nu_T = 1$ peak is locked on to $V=0$, even for (small) non-zero density differences. Despite these indications, however, it is very important to know whether coherent tunneling can give rise to results which are manifestly physically distinct from the corresponding incoherent tunneling results, and, if so, what those results are.

One of such distinctions can be obtained in the d/l_B dependence of the zero-bias tunneling conductance. In Eq.(7), the only dependence of the incoherent tunneling current on d is through the tunneling gap, t. Experimentally, d/l_B is varied usually by changing the magnetic field for the same physical sample (Note that the electron density is also adjusted to maintain the fixed filling factor). In this experimental setup, t should not change much since t depends on the physical distance d, not on d/l_B. Experimentally observed, however, is an abrupt change in the zero-bias tunneling conductance: the zero-bias conductance peak completely disappears after a certain critical distance d_c/l_B[1]. This is the reason why it is so important to develop a quantitative theory to compute the zero-bias conductance peak as a function of d/l_B. We provide such a theory in the next section. We believe that the key to understanding the interlayer coherent tunneling experiment of Spielman *et al.*[1] is its d/l_B dependence, which, as a matter of principle, can distinguish between incoherent single-particle tunneling and the many-body correlation induced coherent effect of bilayer excitonic condensates.

III. DERIVATION OF THE MANY-BODY TUNNELING HAMILTONIAN

In this section, we develop a theory of coherent tunneling which occurs when there is an interlayer coherence in the ground state. An important starting point is that the two layers cannot be regarded as independent systems. Instead, they must be treated as a coherent whole. Since there is no interlayer chemical potential in a single coherent bilayer system, there is no electromotive force within the bilayer system and therefore any current should be induced from outside. So, it is necessary to take into account external leads, as schematically shown in Fig.1. This, of course, makes any quantitative prediction dependent on the way in which bilayer systems are connected to external leads. However, it is still possible to make quantitative predictions on some essential aspects of coherent interlayer tunneling. In particular, we study the d/l_B dependence of the zero-bias tunneling conductance peak. In doing so, we also show that the width of the conductance peak is fundamentally finite even at zero temperature, and it is controlled ultimately by very small, but finite single-particle interlayer tunneling gap t. This makes sense since a finite t is essential to experimentally drive an interlayer current.

Let us begin our quantitative analysis by writing the total Hamiltonian including (i) the interlayer (single-particle) tunneling Hamiltonian, H_t, (ii) the Hamiltonian for the Coulomb interaction between electrons, $H_{\rm Coul}$, (iii) the Hamiltonian describing the left and the right lead, H_L and H_R respectively, and finally (iv) the Hamiltonian for tunneling between leads and the bilayer system, H':

$$\begin{aligned} H &= H_0 + H' + H_R + H_L, && (9)\\ H_0 &= H_t + H_{\rm Coul}, && (10)\\ H_t &= t\sum_m (c^\dagger_{m\uparrow}c_{m\downarrow} + c^\dagger_{m\downarrow}c_{m\uparrow}) \equiv 2tS_x, && (11)\\ \frac{H_{\rm Coul}}{e^2/\epsilon l_B} &= \mathcal{P}_{LLL}\Big(\sum_{i,j\in\uparrow}\frac{1}{r_{ij}} + \sum_{k,l\in\downarrow}\frac{1}{r_{kl}} \\ &\quad + \sum_{i\in\uparrow,k\in\downarrow}\frac{1}{\sqrt{r_{ik}^2+(d/l_B)^2}}\Big)\mathcal{P}_{LLL}, && (12)\\ H' &= \sum_{k,m} T_{R\uparrow}(k,m)[c^\dagger_R(k)c_{m\uparrow} + \text{H.c.}] \\ &\quad + \sum_{p,m'} T_{L\downarrow}(p,m')[c^\dagger_L(p)c_{m'\downarrow} + \text{H.c.}], && (13) \end{aligned}$$

where, as before, the pseudospin representation is used and m is a momentum index of the lowest Landau level. $\mathcal{P}_{LLL}$ is the lowest Landau level projection operator. $T_{R\uparrow}(k,m)$ is the tunneling amplitude between the state with momentum k in the right lead, and the state with m in the top layer of the bilayer system. $T_{L\downarrow}(p,m')$ is similarly defined. H_R and H_L describe external leads as normal Fermi liquids. Note that H_t is related to the pseudospin magneization in the x-direction, S_x:

$$S_x = \frac{1}{2}\sum_m (c^\dagger_{m\uparrow}c_{m\downarrow} + c^\dagger_{m\downarrow}c_{m\uparrow}). \quad (14)$$

Also, note that the internal interlayer current operator is related to the pseudospin magneization in the y-direction, S_y:

$$\hat{J}_{\rm inter} = eti\sum_m (c^\dagger_{m\uparrow}c_{m\downarrow} - c^\dagger_{m\downarrow}c_{m\uparrow}) \equiv 2etS_y. \quad (15)$$

Now that we are interested in the limit of zero $t/(e^2/\epsilon l_B)$, we investigate the possibility of a current operator which survives as $t/(e^2/\epsilon l_B) \to 0$. This operator will, then, measure the spontaneous interlayer current as opposed to the single-particle hopping induced current. To this end, let us consider the following tunneling Hamiltonian:

$$H'_T = H' \frac{1}{E_g - H_0 - H_R - H_L} H', \tag{16}$$

where E_g is the ground state energy of $H_0 + H_R + H_L$. The idea is that, by adding an electron to the top layer and removing another from the bottom layer, H'_T describes the total current flowing through the bilayer system. Note that H'_T is second order in H' since it is the lowest order that can carry the current from one lead to the other lead in the limit of zero $t/(e^2/\epsilon l_B)$.

Normally, the above process would not cause actual interlayer tunneling, but instead it would result in charge build-up. Our bilayer quantum Hall system, however, is special such that it has spontaneous interlayer coherence for small interlayer distances and therefore electrons can move back and forth between the layers even in the limit of zero $t/(e^2/\epsilon l_B)$. The total interlayer current is, of course, zero in the ground state since the back and forth currents cancel each other. Non-zero interlayer currents, however, can be induced by applying a bias voltage which breaks the balance between the back and forth motion in equilibrium.

Once interlayer currents begin to flow as described by the process of H'_T in Eq.(16), the currents can flow steady only when it is carried by the internal interlayer current: $I = \langle \hat{J}_{\rm inter} \rangle$, as dictated by the charge conservation. The exciton condensate can carry the interlayer current by adjusting its interlayer phase difference ϕ. To understand this, it is instructive to consider the ground state wavefunction at $d/l_B \to 0$ in the presence of non-zero interlayer phase:

$$|\psi_{111}(\phi)\rangle = \prod_m (c^\dagger_{m\uparrow} + e^{i\phi} c^\dagger_{m\downarrow})|0\rangle. \tag{17}$$

It can, then, be shown that the ground state in Eq.(17) carries a non-zero interlayer current:

$$\langle \hat{J}_{\rm inter} \rangle = 2et\langle S_y \rangle = 2et\langle S \rangle \sin\phi, \tag{18}$$

where $\langle S \rangle \equiv |\langle \sum_m c^\dagger_{m\uparrow} c_{m\downarrow} \rangle|$.

It is important that there is a maximum current that can be carried by the coherent state:

$$\langle \hat{J}_{\rm inter} \rangle \leq J_{\rm critical} = 2et\langle S \rangle. \tag{19}$$

The existence of the maximum coherent current is analogous to that of the critical current in the superconductivity. This termination of the coherent tunneling current beyond a critical value is the reason why the tunneling conductance has a narrow peak near zero bias. That is, the tunneling current is coherent only within a small window of bias voltage. Once the bias voltage gets larger than a critical value, the tunneling current becomes too large to be carried only by coherent tunneling. The excess current must, then, be carried by incoherent tunneling, which has a low conductance, as argued previously in terms of the formation of the composite fermion sea.

It is important to note that H'_T in Eq.(16) is zeroth order in t whereas, in usual incoherent tunneling, the effect of the tunneling Hamiltonian would be second order in t since the expectation value of S_x in Eq.(11) is also proportional to t. Therefore, for spontaneously coherent systems, the interlayer tunneling conductance does not depend on t in the limit of zero t while the charge conservation condition simply imposes an upper bound on the value of coherent tunneling current, which is proportional to t. In other words, our theory is consistently constructed up to the first order of t, which is contrasted to the second order behavior of the usual theory for incoherent tunneling conductance.

Now, to proceed further, let us examine H'_T in Eq.(16) in detail. Since the bilayer quantum Hall state is incompressible at sufficiently small d/l_B, adding or removing electrons costs a finite energy, Δ,[25] which is equal to either (i) the Coulomb self-energy of a quasiparticle, Δ_C, for the spontaneous coherence occurring when $t \ll e^2/\epsilon l_B$, or (ii) the single-particle tunneling gap, t, for the induced coherence occurring when $t \gg e^2/\epsilon l_B$. While it is possible to investigate both regimes, the spontaneous coherence is particularly interesting, as seen in later sections. It is, however, sufficient at this stage to know that, no matter whether Δ is Δ_C or t, Δ is independent of the momentum index, m, in the lowest Landau level. So we just replace $H_0 + H_R + H_L - E_g$ by Δ.

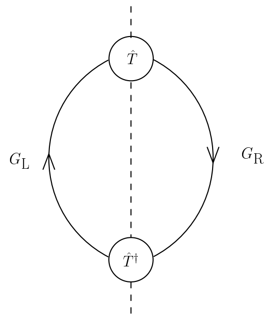
Note that we are able to make the above simplification because H' in Eq.(13) is chosen such that electrons are inserted directly into Landau level eigenstates with momentum indices m. Since each Landau-level eigenstate remains to be a well-defined quasiparticle eigenstate in the presence of the Coulomb interaction, one may take each m state to be a very good approximation to the exact eigenstates of $H_0 + H_R + H_L$ with eigenenergy $E_g + \Delta$ (Note that Δ is a self-energy correction).

Now, we assume that the tunneling amplitudes $T_{R\uparrow}(k, m)$ and $T_{L\downarrow}(p, m')$ are more or less independent of momenta k and p, which is a common practice in tunneling theories where tunneling occurs only within a narrow region of energy near Fermi surface. Keeping only the terms in H'_T relevant for transporting electrons from one lead to the other, we arrive at the following many-body tunneling Hamiltonian:

$$H_T = \sum_{k,p} \left[c^\dagger_R(k) c_L(p) \hat{T}^\dagger + c^\dagger_L(p) c_R(k) \hat{T} \right], \tag{20}$$

where

$$\hat{T} = \frac{1}{\Delta} \sum_m T_{RL}(m) c^\dagger_{m\uparrow} c_{m\downarrow} \tag{21}$$



$\hat{T} = \frac{1}{\Delta_C}\Sigma_m\, T_{RL}(m) c^\dagger_{m\uparrow} c_{m\downarrow}$

FIG. 3: Feynman diagram for coherent tunneling in bilayer quantum Hall systems. The vertex operator $\hat{T}$ contains many-body effects of the exciton condensate. T_{RL} is the tunneling amplitude and Δ_C is the Coulomb self-energy of a quasiparticle.

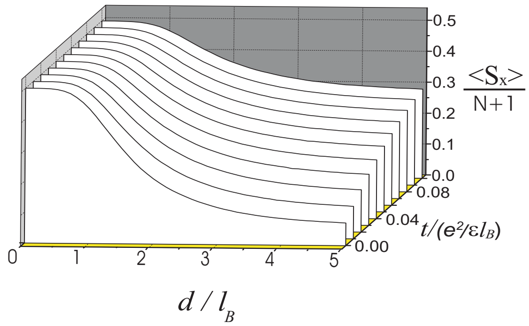


FIG. 4: Interlayer coherence quantified by the pseudospin magnetization, $\langle S_x \rangle$, for the $N = 11$ system. $\langle S_x \rangle$ is computed via exact diagonalization as a function of the interlayer distance, d/l_B, and the single-particle tunneling parameter, $t/(e^2/\epsilon l_B)$. Note that the shape of $\langle S_x \rangle$ is qualitatively similar to the usual magnetization curve as a function of temperature and external magnetic field, which correspond to interlayer distance, d, and single-particle tunneling gap, t, respectively. Also, note that $\langle S_x \rangle$ in the plot shows typical finite-system smearing of a sharp phase transition. The thermodynamic limit is estimated in Fig.5 for $t/(e^2/\epsilon l_B) = 0$.

and $T_{RL}(m) = T_{R\uparrow}(k_F, m) T_{L\downarrow}(k_F, m)$. Based on H_T, the tunneling current operator $\hat{J}$ is defined as follows:

$$\hat{J} = ei \sum_{k,p} \left[c^\dagger_R(k) c_L(p) \hat{T}^\dagger - c^\dagger_L(p) c_R(k) \hat{T} \right]. \qquad (22)$$

Assuming that the tunneling amplitude between leads and the bilayer system is small, one can compute the expectation value of current operator via a conventional first-order S-matrix expansion:

$$I(t) = -i \int_{-\infty}^{t} dt' \langle [\hat{J}(t), H_T(t')] \rangle. \qquad (23)$$

The new aspect of our tunneling theory is the vertex operator, $\hat{T}$, which contains all of many-body effects of the exciton condensate. Eq.(23) can be evaluated further using the Feynman diagram depicted in Fig.3:

$$\begin{aligned} I &= 2e |\langle \hat{T} \rangle|^2 \sum_{k,p} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} A_R(k,\varepsilon) A_L(p, \varepsilon + eV) \\ &\times \; [f(\varepsilon) - f(\varepsilon + eV)] \\ &= 4\pi e^2 D_R D_L |\langle \hat{T} \rangle|^2 V \qquad (24) \end{aligned}$$

where A_R (A_L) is the spectral function of the right (left) lead, $f(\varepsilon)$ is the usual Fermi-Dirac distribution function, and D_R (D_L) is the density of states at the Fermi surface of right (left) lead. Note that we have used $|\langle \hat{T} \rangle|^2$ in Eq.(24) instead of $\langle \hat{T}^\dagger \hat{T} \rangle$ since $|\langle \hat{T} \rangle|^2$ is a much more acute measure of the spontaneous interlayer order in finite-size system diagonalization studies while the two quantities generate identical results in the thermodynamic limit.

Eq.(24) indicates that there is no DC Josephson effect in a conventional sense since the conductance G ($\equiv dI/dV \propto |\langle \hat{T} \rangle|^2$) is finite. It is important to know, however, that the coherent tunneling current is zero if $\langle \hat{T} \rangle = 0$. Remember that $\langle \hat{T} \rangle$ measures the phase coherence between states with various interlayer number differences, $N_{\rm rel}$ (Note that $\hat{T} \propto c^\dagger_{m\uparrow} c_{m\downarrow}$). So, unless the ground state is a coherent linear combination of states with various $N_{\rm rel}$, $\langle \hat{T} \rangle$ is zero, and so is the tunneling current. As mentioned before, this is precisely analogous to the phase coherence between different number eigenstates in superconductivity, which is responsible for the Josephson effect. In this sense, the interlayer tunneling conductance is related to the Josephson effect. However, we emphasize that the conductance is fundamentally finite even at zero temperature. In the next section, we will actually evaluate the interlayer tunneling conductance as a function of d/l_B by using numerical methods. In particular, we will be interested in the normalized conductance since the absolute scale of the conductance is sensitive to sample-specific details such as D_R, D_L and $T_{\rm RL}$.

In concluding this section, we mention that the basic idea in our derivation of the many-body tunneling Hamiltonian is rather similar to that of the linear response theory; if a perturbation is small enough, one can extract various dynamic properties of the biased system from its ground state properties in equilibrium. In our manuscript, we make a connection between the pseudospin order parameter of the ground state and the tunneling conductance of the steady state. This connection can be valid as long as the interlayer current is not too large.

IV. NUMERICAL EVALUATION OF THE COHERENT TUNNELING CURRENT

To study the d/l_B dependence of the tunneling conductance, one has to compute $|\langle \hat{T} \rangle|^2$ in Eq.(24), which

can be further reduced as follows:

$$|\langle\hat{T}\rangle|^2 = \frac{\langle S_x\rangle^2}{\Delta^2}\left|\frac{1}{N}\sum_m T_{\rm RL}(m)\right|^2, \qquad (25)$$

where N is the total number of electrons. Here, we have simplified the evaluation of the many-body tunneling matix element, $\hat{T}$, by assuming that the interlayer phase coherence is uniformly obtained throughout the system; in other words, the interlayer coherence order parameter $\langle c^\dagger_{m\uparrow}c_{m\downarrow}\rangle$ is independent of m.

In this situation, the m-dependence of tunneling amplitudes between leads and correspoding adjacent layers, $T_{\rm RL}(m)$, affects only the overall magnitude of the tunneling conductance, leaving its d/l_B dependence unchanged. Note that effectively what we are assuming is that the lead does not alter important bilayer properties such as the interlayer distance and the single-particle tunneling amplitude. Now, since $\langle c^\dagger_{m\uparrow}c_{m\downarrow}\rangle$ is independent of m, $\langle c^\dagger_{m\uparrow}c_{m\downarrow}\rangle = \langle S_x\rangle/N$. S_x $[=\sum_m(c^\dagger_{m\uparrow}c_{m\downarrow}+c^\dagger_{m\downarrow}c_{m\uparrow})/2]$ can be regarded as an order parameter of the exciton condensate. It is natural to assume that $\sum_m T_{\rm RL}(m)/N$ does not depend on d/l_B since $T_{\rm RL}$ is only dependent on the hopping amplitude between lowest-Landau-level states in a given layer and plane-wave states in the external lead connected to the layer. The d/l_B dependence of the conductance is, therefore, solely determined by $\langle S_x\rangle^2/\Delta^2$.

Figure 4 shows the three-dimensional plot of $\langle S_x\rangle$ as a function of the interlayer distance, d/l_B, and the single-particle tunneling gap, $t/(e^2/\epsilon l_B)$. $\langle S_x\rangle$ in the plot was computed via exact diagonalization for the $N=11$ system. Note that, when computing $\langle S_x\rangle$ in finite systems, it is very important to take into account fundamental fluctuations in $N_{\rm rel}$; the true ground state is a coherent, linear combination of states with various $N_{\rm rel}$. Technical details can be found in the literature[26,27,28]. Also, note that $\langle S_x\rangle$ in the plot shows typical finite-system smearing of a sharp phase transition. It is interesting that the shape of $\langle S_x\rangle$ is qualitatively similar to the typical magnetization curve as a function of temperature and external magnetic field, which respectively correspond to d/l_B and $t/(e^2/\epsilon l_B)$.

Figures 4 and 5 show that, for $d < d_C \simeq 1.4-1.7 l_B$, $\langle S_x\rangle$ does not vanish even in the limit of $t/(e^2/\epsilon l_B)\rightarrow 0$, which is, by definition, the evidence for the spontaneous order. Since $\langle S_x\rangle$ becomes zero for $d > d_C$ in the thermodynamic limit, it is natural that the tunneling conductance becomes zero as well, as dictated by Eqs.(24) and (25). On the other hand, if induced by the single-particle tunneling alone (or, if $t \gg e^2/\epsilon l_B$), the coherence does not completely vanish at a finite d/l_B, nor does the corresponding coherent tunneling conductance peak. This is not consistent with experiment where the conductance peak disappears at a finite critical interlayer distance.

So, from now on, we focus on the limit of very small single-particle tunneling gap, $t/(e^2/\epsilon l_B)\rightarrow 0$, where the coherence is spontaneous. Fig.5 plots $\langle S_x\rangle^2$ as a function of d/l_B for various particle numbers when $t/(e^2/\epsilon l_B)\rightarrow 0$. While the accurate estimate of the thermodynamic limit of $\langle S_x\rangle^2$ is difficult due to numerical uncertainties, it is reasonable to assume that the true thermodynamic limit lies within the shaded region in Fig.5. As typical in finite-system calculations, inflection points are taken to be the finite-system signature for the true critical point in the thermodynamic limit.

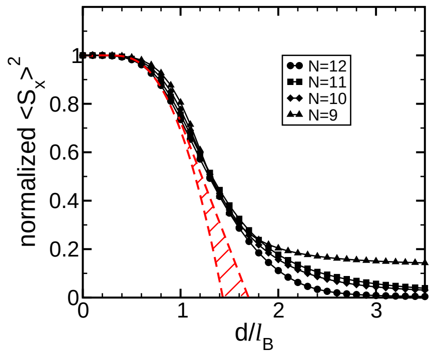


FIG. 5: Normalized expectation value of the condensate order parameter $\langle S_x\rangle^2$ in the limit of $t/(e^2\epsilon l_B)\rightarrow 0$. In order to incorporate finite-system corrections, $\langle S_x\rangle$ is normalized in such a way that it is divided by $(N+1)/2$ for N odd and by $\sqrt{N(N+2)}/2$ for N even[26]. N is the total number of electrons in finite-system exact diagonalization studies. The shaded region indicates the thermodynamic-limit estimate for $\langle S_x\rangle^2$.

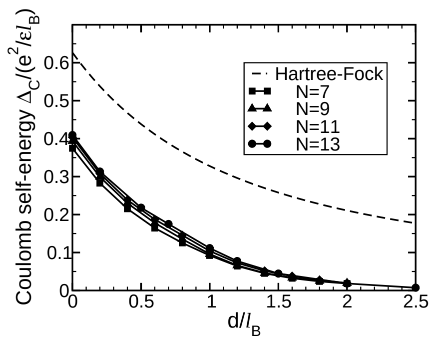


FIG. 6: Energy cost of adding (removing) an electron into (from) the ground state of the bilayer quantum Hall system. Note that this energy cost is essentially the self-energy of a quasiparticle due to the Coulomb interaction. The Coulomb self-energy is computed via exact diagonalization (for various finite systems) as a function of interlayer distance, d/l_B. For comparison, also plotted is the self-energy estimate obtained in the Hartree-Fock approximation.

In Fig.6, we compute the Coulomb energy cost of creating a quasiparticle, Δ_C, which is the only energy cost when $t/(e^2/\epsilon l_B)\rightarrow 0$. That is, $\Delta=\Delta_C$ in Eq.(25). Δ_C is computed via exact diagonalization in finite systems. It is interesting to note that, for general values of d/l_B, the exact Δ_C is much lower than the estimate obtained in the Hartree-Fock approximation[29,30].

Finally, in Fig.7, our theoretical estimate for the nor-

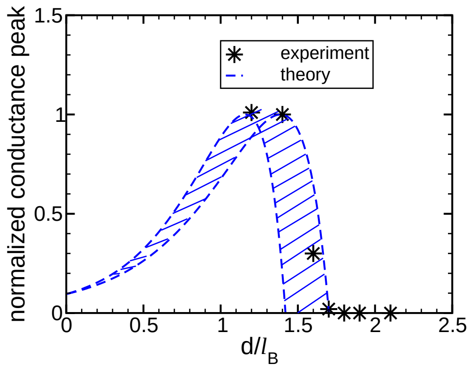


FIG. 7: Normalized interlayer tunneling conductance peak (the shaded region in the plot) as a function of d/l_B in comparison with experimental data from Spielman *et al.*[1]. We define the normalized conductance as the conductance divided by its maximum value as a function of d/l_B. The shaded region is obtained from the thermodynamic estimate of $\langle S_x \rangle^2$ in Fig.5.

malized interlayer tunneling conductance near zero bias, *i.e.* $\langle S_x \rangle^2/\Delta_C^2$, is compared with experimental data of Spielman *et al.*[1]. We define the normalized conductance as the conductance divided by its maximum value as a function of d/l_B. The shaded region in the plot is the thermodynamic estimate. To the best of our knowledge, the direct comparison with experimental data for the conductance peak is made here for the first time. It is interesting that our theory predicts that the conductance peak decreases as d/l_B decreases below roughly 1.2. Remember that the decrease in the conductance peak at small d/l_B is due to the increase in Δ_C while the pseudospin magnetization is saturated.

We would like to emphasize that our theoretical estimate of the conductance peak has only one fitting parameter: the overall prefactor dependent on sample specifics. Considering that the only fitting parameter is an overall scale factor, we argue that the agreement with experiment is not too bad. Specifically, our theory provides a reasonably accurate estimate for the critical distance at which the conductance peak disappears. While the critical distance was estimated previously, its determination has been based on indirect evidence: it was determined either (i) by studying the collapse of the overlap between the Halperin's (1,1,1) state and the exact ground state[31,32], or (ii) by studying the collapse of the low-energy excitation in a time-dependent Hartree-Fock approximation[30] or (iii) by approaching the critical point from the incoherent side and studying the effect of the single-particle tunneling[33]. Our theory, on the other hand, is a direct study of the tunneling conductance itself, as measured experimentally, rather than a consideration of an energy gap collapse or similar purely theoretical constructs associated with the quantum phase diagram.

Aside from the direct estimation of the critical distance, our theory predicts a rather steep rise of the peak height as d/l_B is reduced from the critical value, as seen in Fig. 7. This steep rise cannot be explained by the collapse of the pseudospin magnetization, $\langle S \rangle$, alone; it requires a detailed consideration of the coherent tunneling process.

In the preceding section, we have shown that there is a critical interlayer current allowed without breaking the phase coherence. Consequently, for a sufficiently large bias voltage, the coherent interlayer current should be cut off. We have argued that this is the reason why the interlayer conductance has a shape of the sharp peak near zero bias. Now, we would like to discuss what determines the width of the conductance peak. By equating Eqs.(19) and (24), one can show that, while the proportionality constant strongly depends on sample-specific details, the width of conductance peak is proportional to t. Due to the strong dependence on sample details, the accurate estimation of the critical current is beyond the scope of this paper. It is, however, encouraging to find that typical width of the conductance peak ($\sim 10\mu eV$) is in the similar order as the single-particle tunneling gap[1,8]. Our finding that the width of the conductance peak is proportional to the tunneling amplitude t shows that the existence of a finite symmetry breaking term, i.e. a nonzero t, is essential for the experimental observations in Spielman *et al.*[1]

V. PROPOSAL FOR THE JOSEPHSON EFFECT IN CONTERFLOWING CURRENT MEASUREMENTS

Until now, we have studied interlayer tunneling in a single bilayer system, which, we showed, is not an exact analog of the Josephson effect. In this section we propose a much more direct analog of the real Josephson effect. Since the coherent particle-number fluctuation in superconductivity holds a parallel with the coherent fluctuation of interlayer number differences in bilayer quantum Hall systems, it is natural to expect that the corresponding Josephson effect for the bilayer quantum Hall systems must manifest itself in current measurements associated with interlayer number difference. Such measurements are counterflowing current measurements[34,35].

To be specific, let us consider a weakly coupled set of two bilayer quantum Hall systems, say A and B (four layers altogether), separated by a lateral tunneling barrier. We further consider the situation where there is no cross tunneling between the top layer of the system A and the bottom layer of the system B. Tunneling occurs either (i) between the top and the bottom layer of the same system or (ii) between the same layers of the system A and B. A schematic diagram is shown in Fig.8.

The configuration similar to the one depicted in Fig.8 has been, in fact, proposed previously[36]. The important twist in this paper is the application of the interlayer current flowing vertically through the top and the bottom layer of one of the bilayer systems. The purpose is

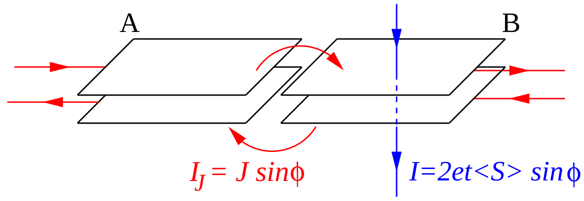


FIG. 8: Schematic diagram for the proposed Josephson effect in counterflowing current measurements in a coupled set of two bilayer quantum Hall systems.

to induce a non-zero interlayer phase difference in one of the bilayer systems while the other system has none. Note that, in bilayer quantum Hall systems, the phase of the condensate is controlled via vertical interlayer current flow which affects the internal degree of freedom of the condensate. It is interesting that there is no similar way of controlling the phase in usual superconductors.

Specifically, when the applied interlayer current is I, a non-zero interlayer phase is induced so that

$$\sin\phi = I/2et\langle S\rangle, \tag{26}$$

where $\langle S\rangle$ is defined in Eq.(18). Our prediction then is that, due to the analogy with the Josephson effect in superconductivity, there will be two counterflowing currents with one flowing between the top layers of the system A and B, and with the other flowing between the bottom layers; the Josepson effect for the neutral, total current in our system is given by

$$I_J = J\sin\phi = \frac{J}{2et\langle S\rangle} I, \tag{27}$$

where the proportionality constant, J, is dependent on tunneling parameters between the system A and B. Note that, when I_J is the current flowing between the top layers, the current of $-I_J$ flows between the bottom layers. The net current is zero, but it may be possible to measure these two counterflowing currents individually.

Let us mention what conditions should be satisfied for our prediction to be realized. In the above, we have already mentioned that the cross tunneling between different layers of the system A and B is assumed to be negligible. This is due to the fact that the cross tunneling cannot carry the supercurrent. A maybe more important condition, however, is that the applied current, I, should not exceed the critical current since, otherwise, the coherence will be lost. The experiment described in the above, therefore, can be performed only in a very narrow range of the applied current, which, in turn, corresponds to a very narrow bias-voltage range.

VI. CONCLUSION

It has been shown in the preceding sections that the interlayer tunneling conductance peak near zero bias is strongly enhanced, but fundamentally finite even at zero temperature. The reason why the interlayer conductance peak is not intrinsically infinite can be attributed to the fact that the experimental setup measuring the interlayer tunneling conductance is not a setup for the true Josephson effect in the bilayer exciton condensate. To substantiate this idea, we have computed the height of the zero-bias interlayer conductance peak as a function of interlayer distance, which is, then, compared with experiment. Considering that the only fitting parameter is an overall scale factor which strongly depends on sample specifics, the agreement with experiment is reasonable. Specifically, our theory provides a reasonably accurate estimate for the critical distance in which the conductance peak disappears. Our theory is contrasted from previous theories in that it determines the critical distance by explicitly computing the interlayer tunneling conductance as a function of interlayer distance. Thus, our theory is a transport theory for the tunneling conductance, which is precisely what is measured experimentally.

While one of the main goals of this paper is to develop a concrete theory of the coherent tunneling conductance, it is another goal of this paper to make a sharp distinction between the spontaneous coherence due to the Coulomb interaction and the induced coherence due to the single-particle tunneling. In this respect, the detailed computation of the conductance peak height as a function of d/l_B is also very important because it offers such sharp distinction; if the coherence is due to the Coulomb interaction, the conductance peak will completely disappear at a finite d/l_B, as discussed in Sec.II.

We have also discussed the similarities and the differences between the bilayer quantum Hall interlayer coherence phenomena and the related collective coherent physics manifested in magnetic phenomena and in superconducting Josephson effect. In particular, we have proposed an experimental setup for the Josephson effect in couterflowing current measurements in a coupled set of two bilayer quantum Hall systems, which is a precise analogy with the real Josephson effect in superconductivity.

VII. ACKNOWLEDGEMENT

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